The Diffusion Flame as a Singular Perturbation Problem

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SUMMARY
Solutions of the chemical species' conservation equations are shown to involve two small parameters and to give rise to singular perturbation problems. General results are derived for arbitrary flame geometry and a criterion for the retention in a solution of both parameters is established.

1. Introduction

The relation which expresses conservation of mass of the reactant species (**F**, fuel and **X**, oxidant) in a combustible mixture can be written in the form

$$\varepsilon \mathbf{V} \cdot \{\rho \gamma_\alpha \boldsymbol{u} - \rho \mathscr{D} \mathbf{V} \gamma_\alpha\} = -(\rho/\tau)\{c_X^{\nu_X} c_F^{\nu_F} - (1 - c_X - c_F)^{\nu_P} \delta\}, \qquad \alpha = X, F, \tag{1}$$

where the variables (all dimensionless) have the following meanings; ρ is the density, c_α is the mass fraction of species α, τ is the characteristic reaction time, $\boldsymbol{u}$ is the flow velocity vector, $\mathscr{D}$ is the diffusion coefficient and the ν_α quantities ($\alpha = X, F, P$) are the stoichiometric integers in the reaction equation

$$\nu_F \mathbf{F} + \nu_X \mathbf{X} \rightleftharpoons \nu_P \mathbf{P}. \tag{2}$$

P is the product species, γ_α is defined so that

$$\gamma_\alpha = c_\alpha / \nu_\alpha W_\alpha, \tag{3}$$

where W_α is the molecular weight of species α, δ is the equilibrium constant for the reaction and ε is the ratio of a typical (dimensional) reaction time to a typical (also dimensional) diffusion time. The gradient operator is denoted by $\mathbf{V}$.

The combustion situation is distinguished by the much greater availability of energy states to the **F** and **X** reactants when they are combined in a **P** molecule (provided that the temperatures are not excessively high) and accordingly we may take it that

$$\delta \ll 1.$$

By hypothesis, any situation which is diffusion-controlled must have

$$\varepsilon \ll 1.$$

It follows that this theoretical model of a diffusion flame is characterised by the appearance of two small parameters, at least as far as the species conservation equation is concerned. The remaining conservation equations will evidently introduce other parameters, like the Reynolds number, Prandtl number, and so on, but it will not be necessary here to become directly involved with any of these other aspects of the flow system.

It is, indeed, a primary purpose of this account of the diffusion flame to show how much can be learnt from equation (1) alone by making use of singular perturbation theory. Actual combustion situations have more complicated kinetics than (2) and must involve multi-component diffusion (equation (1) embodies a Fick's-law description of the mass diffusion processes) and these matters have been considered in previous articles [1, 2, 3, 4, 5]. The present simplified model illustrates some of the foundations on which these theories have been constructed and hence may be of some pedagogical interest. Of course such a simple model has

already been used by a number of authors [6, 7, 8, 9, 10, 11] but, just as with the previous references, always in the context of a specific geometry. The role played by flame geometry will be shown here by analysing a very general situation. This general approach also provides an opportunity to display matched asymptotic expansion [see e.g. 12] theory applied to a problem with two small parameters. The solutions of (1) are singular in the limit when both ε and δ vanish, but overall regularity can be restored by resurrecting ε as a small non-vanishing parameter while keeping δ zero, or *vice versa*; the solutions are quite different in each case. A criterion for retention of both parameters emerges from the analysis, which is of some additional interest in that location of the region of non-uniformity in the limit is not known *a priori*.

Although it may not be possible to do so in all cases [see e.g. 4, 10] it helps in present circumstances to treat ρ, $\mathscr{D}$, τ and $\boldsymbol{u}$ as given functions of position. Then solutions of (1) are required in a domain D, subject to a requirement of continuity of γ_α (for physical reasons) and to suitably imposed boundary value data. Evidently (1) is a partial differential equation of elliptic type (the highest order derivatives appear in the form $-\varepsilon\rho\mathscr{D}\nabla^2\gamma_\alpha$, where ∇^2 is the Laplacian operator) so that such data is required everywhere on the boundary ∂D of D; it may be of mixed form in the sense, for example, that γ_α may be given on part of ∂D while its normal derivative is then given on the remainder.

Before proceeding it is important to note that subtraction of equation (1) with $\alpha = \mathrm{F}$ from (1) with $\alpha = \mathrm{X}$ gives

$$\boldsymbol{\nabla}\cdot\{\rho\boldsymbol{u}(\gamma_\mathrm{X}-\gamma_\mathrm{F})-\rho\mathscr{D}\boldsymbol{\nabla}(\gamma_\mathrm{X}-\gamma_\mathrm{F})\} = 0\,, \tag{4}$$

which is *independent of both ε and δ*. Such relations are always available, being in the nature of "atom" (or "basic element") conservation laws.

2. Singular Solutions

Consider the outer limit

$$\varepsilon\downarrow 0\,,\quad \delta\downarrow 0\,;\quad \psi \text{ fixed}\,, \tag{5}$$

where ψ stands for all of the variables (dependent and independent) in (1) and (4). The first equation reduces to

$$c_\mathrm{X}^{\gamma_\mathrm{X}} c_\mathrm{F}^{\gamma_\mathrm{F}} = 0 \tag{6}$$

and the second is unchanged.

The only non-trivial solution of (6) is

$$\begin{aligned}c_\mathrm{F} = 0\,,\quad c_\mathrm{X} \neq 0 \text{ in } D_\mathrm{X}\,,\\ c_\mathrm{F} \neq 0\,,\quad c_\mathrm{X} = 0 \text{ in } D_\mathrm{F}\,,\end{aligned} \tag{7a}$$

where D_X and D_F are non-overlapping domains which together must make up the whole of D. Evidently D_X and D_F will be separated by a surface S which has oxidant on one side of it (say "outside") and fuel on the other (say "inside"). Continuity of c_α (or γ_α) will make it necessary for $c_\mathrm{X} \to 0$ as the position vector $x \to x_S$ (value of x on S) from within D_X while $c_\mathrm{F} \to 0$ as $x \to x_S$ from within D_F.

It now follows from (4), etc. that

$$\boldsymbol{\nabla}\cdot\{\rho\boldsymbol{u}c_\alpha - \rho\mathscr{D}\boldsymbol{\nabla}c_\alpha\} = 0,\qquad \alpha = \mathrm{X}, \mathrm{F}\,, \tag{7b}$$

with boundary value data given on the boundary ∂D_α of D_α; such data must include the prescription, $c_\alpha \to 0$ as $x \to x_S$, but will otherwise be as specified on that portion of ∂D which bounds D_α. Clearly the problems for c_α in D_α will be properly posed, provided the location (geometry) of S is known. Solutions for c_α in D_α will have continuous first and second derivatives but it is clear from (7) that the first normal derivative of (say) c_X will change discontinuously on crossing S from D_X into D_F, with a similar behaviour of c_F on crossing S in the other direction. A condition linking solutions across S will suffice to determine its location and such a condition is derivable

from integration of (4) across S along any normal co-ordinate direction. If the co-ordinate n is positive outside S and negative inside it, it readily follows from continuity of the dependent variables that

$$\frac{\partial}{\partial n} \gamma_X(n\downarrow 0) = - \frac{\partial}{\partial n} \gamma_F(n\uparrow 0) \tag{8}$$

and this is the required condition. It has the interesting physical interpretation of stoichiometric rates of diffusion of fuel and oxidant into the "flame sheet" S [13].

If c_α is specified to be positive on all parts of ∂D_α other than S (where it is zero) the maximum principle for linear elliptic differential equations [14] can be used to demonstrate that c_α will be positive and will have no maxima or minima within D_α (observe that there is no term in c_α in (7b) since $\mathbf{V} \cdot \rho\mathbf{u}$ vanishes by reason of conservation of mass). In such circumstances it must always be possible to find an S within D. If it is the normal gradients of c_α which are specified on those parts of ∂D_α other than S, then there is no *a priori* guarantee that condition (8) can be met with a positive c_α in D_α; in such cases any S defined by (8) would not be within D and the solutions would not be physically acceptable (they correspond to the case of too small an injection rate through a solid surface, for example).

We have now established sufficient information about the limiting solution to recognize that it is singular on S insofar as the second normal derivatives of c_α (or γ_α) become unbounded there. Thus the double limit of vanishing ε and δ is properly identifiable as an outer limit and the solutions (7) should be re-interpreted as outer estimates of c_α to $O(1)$ in the limit as both ε and δ, separately, approach zero. The necessity to, first, correct for the non-uniformities at S and, second, to find the first significant outer estimate of c_X within D_F and c_F within D_X, for they are not truly zero there being, rather, $o(1)$ so far as is known at present, means that it is convenient to deal in terms of an orthogonal co-ordinate system with co-ordinates normal (i.e. n) and tangential (i.e. s and t) to S. In such a system S is defined to be the surface on which n is zero (as in (8)).

It must also be observed that (4) has a solution in D which it is now convenient to write in the form

$$\gamma_X - \gamma_F = \mathscr{F}(n, s, t). \tag{9}$$

It can be presumed that $\mathscr{F}$ is calculable since specification of data for c_X and c_F separately on ∂D obviously implies that data for $\gamma_X - \gamma_F$ is likewise given on this surface. It must not be presumed that $\mathscr{F}(n, s, t)$ necessarily represents the solution of (7) in D_X and D_F. This can only be strictly so if c_X vanishes exactly on the part of ∂D_F which coincides with ∂D, with a similar condition on c_F and ∂D_X. It will however emerge that this presumption about the general form of $\mathscr{F}$ is implicit in the solutions which are adopted below, at least to a sufficient order of accuracy. To the extent that $\mathscr{F}$ represents a solution of (4) in the domain D it must be independent of ε and δ, be $O(1)$, and have continuous derivatives up to and including the second; we infer that $\mathscr{F}$ must certainly possess a Taylor series in the regions D_X and D_F adjacent to S.

Finally in this section it is observed that (1) in terms of the orthogonal n, s, t system has the form

$$\varepsilon\left\{\frac{\rho\mathscr{D}}{h_n h_s h_t}\left[\frac{\partial}{\partial n}\left(\frac{h_s h_t}{h_n}\frac{\partial\gamma_\alpha}{\partial n}\right) + \frac{\partial}{\partial s}\left(\frac{h_n h_t}{h_s}\frac{\partial\gamma_\alpha}{\partial s}\right) + \frac{\partial}{\partial t}\left(\frac{h_n h_s}{h_t}\frac{\partial\gamma_\alpha}{\partial t}\right)\right] + \text{o.t.}\right\} =$$

$$- (\rho/\tau)\{c_X^{\nu_X} c_F^{\nu_F} - (1 - c_X - c_F)^{\nu_P}\delta\}, \qquad \alpha = X, F, \tag{10}$$

where o.t. stands for other terms which involve only first derivatives of γ_α and which are not of any direct concern for present purposes. The quantities h_i $(i = s, n, t)$ are scale factors which are in general, functions of n, s and t; the question of their order of magnitude will be raised in appropriate parts of the analysis which follows.

3. Solutions for $\varepsilon = 0$, $\delta \downarrow 0$

Making ε zero in (1) (or 10) gives

$$c_X^{v_X} c_F^{v_F} = (1 - c_X - c_F)^{v_P} \delta , \tag{11}$$

while (9) gives the necessary second relation to find c_X and c_F. Flames with structures consistent with (11) are called equilibrium-broadened flames. Although (9) and (11) together may be taken to represent the complete solution, at least to the extent that simple substitution from (9) into (11) gives an algebraic equation for c_α, it is nevertheless much more revealing to define outer and inner series, respectively, as follows (N.B. $\alpha = X$, F only):

$$c_\alpha(n, s, t : \varepsilon, \delta) \simeq c_\alpha(n, s, t : 0, \delta) \sim \sum_{m=1}^{\infty} \left\{ \begin{matrix} f_{ev}^{(m)}(\delta), n > 0 \\ g_{e\alpha}^{(m)}(\delta), n < 0 \end{matrix} \right\} c_{e\alpha}^{(m)}(n, s, t) , \tag{12}$$

where all $c_{e\alpha}^{(m)}(n, s, t)$ are $O(1)$ in $|n| > 0$ and $f_{e\alpha}^{(m+1)}/f_{e\alpha}^{(m)}$, $g_{e\alpha}^{(m+1)}/g_{e\alpha}^{(m)}$ are $o(1)$; similar series hold for γ_α (replace c by γ):

$$c_\alpha(n, s, t : 0, \delta) \sim \sum_{m=1}^{\infty} F_{e\alpha}^{(m)}(\delta) \mathscr{C}_{e\alpha}^{(m)}(N_e, s, t) , \tag{13a}$$

$$N_e = n/\delta^{b_e} , \qquad b_e < 0 , \tag{13b}$$

where all $\mathscr{C}_{e\alpha}^{(m)}(N_e, s, t)$ are $O(1)$ when the stretched normal variable N_e is $O(1)$, and $F_{e\alpha}^{(m+1)}/F_{e\alpha}^{(m)}$ is $o(1)$; similar series hold for γ_α (replace $\mathscr{C}$ by Γ when c is replaced by γ). From the work of section 2 it is clear that

$$f_{eX}^{(1)}(\delta) = 1 = g_{eF}^{(1)}(\delta) , \tag{14}$$

while all other $f_{e\alpha}^{(m)}(\delta)$, $g_{e\alpha}^{(m)}(\delta)$ and $F_{e\alpha}^{(m)}(\delta)$ must be $o(1)$.

Substituting (12) into (9) and (11), and making $\delta \downarrow 0$ with n, s, t fixed shows, again from section 2, that

$$\gamma_{eX}^{(1)} H(n) - \gamma_{eF}^{(1)} H(-n) = \mathscr{F}(n, s, t) , \tag{15a}$$

where $H(n)$ is the unit step function $(= 1, n > 0; = 0, n < 0)$: it is inferred that

$$\mathscr{F}(|n| \to 0, s, t) \to 0 . \tag{15b}$$

The inner series may now be supposed to have first significant gauge functions which are as follows;

$$F_{eX}^{(1)} = \delta^{a_e} = F_{eF}^{(1)} , \qquad a_e > 0 . \tag{16}$$

Substituting (16) into (11), and making $\delta \downarrow 0$ with N_e, s, t fixed, shows that the only non-trivial value for a_e is

$$a_e = 1/(v_X + v_F) \leq 1/2 . \tag{17}$$

It follows that

$$\mathscr{C}_{eX}^{(1)v_X} \mathscr{C}_{eF}^{(1)v_F} = 1 = (v_X W_X \Gamma_{eX}^{(1)})^{v_X} (v_F W_F \Gamma_{eF}^{(1)})^{v_F} . \tag{18}$$

The exact solution (9) yields

$$\Gamma_{eX}^{(1)} - \Gamma_{eF}^{(1)} + \ldots = \delta^{-a_e} \mathscr{F}(N_e \delta^{b_e}, s, t) = \delta^{b_e - a_e} \mathscr{F}_n(0, s, t) N_e + \ldots , \tag{19}$$

where the dots represent higher order terms in the asymptotic and Taylor expansions, and $\mathscr{F}_n$ stands for $\partial \mathscr{F}/\partial n$.

Evidently

$$b_e = a_e \tag{20}$$

$$\Gamma_{eX}^{(1)} - \Gamma_{eF}^{(1)} = \mathscr{F}_n(0, s, t) N_e . \tag{21}$$

The first inner problem, to find $\Gamma_{eX}^{(1)}$, is now seen to be simpler than the original problem posed by (9) and (11). Combining (21) with (18) shows, for example, that

$$\left(v_X W_X \Gamma_{eX}^{(1)}\right)^{v_X} \left\{v_F W_F \left[\Gamma_{eX}^{(1)} - \mathscr{F}_n(0, s, t) N_e\right]\right\}^{v_F} = 1 , \tag{22}$$

which gives N_e as a function of $\Gamma_{eX}^{(1)}$ at once; the "inverse" inner problem is thus easily solved. Asymptotic (i.e. $|N_e| \to \infty$) forms for $\Gamma_{eX}^{(1)}$ are also readily available from (22); thus

$$\Gamma_{eX}^{(1)} \sim \mathscr{F}_n(0, s, t) N_e + \left(v_F W_F\right)^{-1} \left\{v_X W_X \mathscr{F}_n(0, s, t) N_e\right\}^{-v_X/v_F} + \dots, \quad N_e \to \infty , \tag{23}$$

$$\Gamma_{eX}^{(1)} \sim \left(v_X W_X\right)^{-1} \left\{-v_F W_F \mathscr{F}_n(0, s, t) N_e\right\}^{-v_F/v_X} + \dots, \quad N_e \to -\infty , \tag{24}$$

Evidently $\delta^{a_e} \Gamma_{eX}^{(1)}$ matches with $\gamma_{eX}^{(1)}$ in D_X, since (23) shows that the first term in $\delta^{a_e} \Gamma_{eX}^{(1)}$ (N_e, s, t) as $\delta \downarrow 0$, with n (>0) fixed, is equal to $\mathscr{F}_n(0, s, t) N_e \delta^{a_e}$ (because of (13b) and (17)), while the first term of the outer series for γ_X in D_X as $\delta \downarrow 0$, with $N_e(>0)$, s, t, fixed, is equal to $\mathscr{F}_n(0, s, t) N_e \delta^{b_e}$; these two quantities are the same by reason of (20). It is interesting to note that matching has arisen naturally from the assumed nature of the outer and inner series, and has not had to be imposed. This is because the matched asymptotic expansion method is here being used to solve algebraic equations, rather than the differential equations which are more usually met with. Examining (24) in the light of matching requirements [12] makes it apparent that

$$g_{eX}^{(1)}(\delta) = \delta^{1/v_X} , \tag{25}$$

but it is even more readily apparent from (11), and the fact that c_F is $O(1)$ in D_F, that the gauge function $g_{eX}^{(1)}$ must behave as in (25). This is another example of the automatic nature of the matching in this particular algebraic problem. The form of the function $\gamma_{eX}^{(1)}$ in D_F will also follow readily from (11), with $\gamma_{eF}^{(1)}$ found from (15a) ($n<0$). Evidently the whole first-significant behaviour of c_X in D follows from such considerations, and the same will be true for c_F.

On the presumption that all of the scale factors h_i ($i=s, n, t$) and their first derivatives are $O(1)$ as $\delta \downarrow 0$, it can be seen that the left-hand side terms in (10) are $O(\varepsilon \delta^{-a_e})$ in the inner region. This follows from the fact that the double n-derivatives are $O(\delta^{-a_e})$, while all other terms there are $O(1)$, by hypothesis. It follows that (11) is a valid approximation to (10), when both $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, only if $\varepsilon \delta^{-1-a_e}$ is $o(1)$. If this condition is met then a uniformly valid approximation to the concentration field has been found without having to restore the highest-derivative terms in the differential equation, as one usually expects to have to do in a singular perturbation problem.

Under conditions which make the results of this section valid it can be seen that c_X and c_F are affected by flame geometry only through the intervention of the solution $\mathscr{F}(n, s, t)$.

4. Solutions for $\varepsilon \downarrow 0$, $\delta = 0$

Putting δ equal to zero in (10) leads to the equation

$$\varepsilon \left\{ \frac{\rho \mathscr{D}}{h_n h_s h_t} \frac{\partial}{\partial n}\left(\frac{h_s h_t}{h_n} \frac{\partial \gamma_\alpha}{\partial n} \right) + \text{o.t.} \right\} = -(\rho/\tau) c_X^{v_X} c_F^{v_F}, \qquad \alpha = X, F , \tag{26}$$

where o.t. now stands for the "other terms" mentioned below (10 augmented by the two second derivatives in s and t. A flame whose structure is determined by (26) is said to be reaction-broadened. Just as in the previous section, where ε was made to equal zero with $\delta \downarrow 0$, outer and inner series can be defined. With the difference that the approximation (see (12)) now reads

$$c_\alpha(n, s, t : \varepsilon, \delta) \simeq c_\alpha(n, s, t : \varepsilon, 0) ,$$

the series defined in (12) and (13) can be used in the present case with the modifications that all quantities which carry a subscript e in section 2 shall here have e replaced by r and that the parameter ε shall replace δ. Then all of the results, remarks and hypotheses from (12) through to (16) apply otherwise unchanged. In particular the $O(1)$ outer solution (see (14), (15)) is exactly the same, since $\mathscr{F}(n, s, t)$ must be the same in both situations.

However a_r will not be given by (17); it must be found from (26), as follows. Let it be assumed that the scale factors h_i $(i = n, s, t)$ together with their derivatives, are all $O(1)$ quantities; then (26) in the inner region (where the normal co-ordinate N_r is equal to n/ε^{b_r}) has the form

$$\varepsilon^{1 + a_r - 2b_r} (\rho \mathscr{D}/h_n^2)_S \frac{\partial^2}{\partial N_r^2} \Gamma_{r\alpha}^{(1)} + \ldots = (\rho/\tau)_S \varepsilon^{a_r(\nu_X + \nu_F)} (\mathscr{C}_{rX}^{(1)} + \ldots)^{\nu_X} (\mathscr{C}_{rF}^{(1)} + \ldots)^{\nu_F} , \tag{27}$$

where the dots represent higher order terms. The quantities in ( )$_S$ are to be given their values on the sheet S, where n is zero, and it is important to note the implication that such quantities must be functions of s and t alone.

Now results (19), (20) and (21) must also apply here (with subscript e replaced by subscript r and δ by ε), so that the only non-trivial scalings of n and c_α (i.e. which do not either reduce (27) to an outer-region form like (6), or give an unmatchable solution for $\Gamma_{r\alpha}^{(1)}$, namely $\Gamma_{r\alpha}^{(1)}$ proportional to N_r plus a constant) must be

$$a_r = b_r = 1/(1 + \nu_X + \nu_F) \leqq 1/3 . \tag{28}$$

The equation satisfied by $\Gamma_{rX}^{(1)}$ for example, is therefore

$$\mathscr{L}^2 \frac{\partial^2}{\partial N_r^2} \Gamma_{rX}^{(1)} = \Gamma_{rX}^{(1)\nu_X} \{ \Gamma_{rX}^{(1)} - \mathscr{F}_n(0, s, t) N_r \}^{\nu_F} , \tag{29a}$$

where

$$\mathscr{L}^2 = (\tau \mathscr{D}/h_n^2)_S / (\nu_X W_X)^{\nu_X} (\nu_F W_F)^{\nu_F} , \tag{29b}$$

and $\Gamma_{rF}^{(1)}$ has been eliminated by using the appropriately modified version of (21). Observe that $\mathscr{L}$ is a function of s and t only. Any solution of (29) must match properly with the outer solutions for c_X (or γ_X), which are evidently

$$c_X \sim \mathscr{F}(n, s, t) + o(1) , \qquad n > 0 ,$$
$$\sim o(1) , \quad n < 0 .$$

It is best to deal with normal derivatives of c_X, rather than c_X itself, whence it is readily shown that matching leads to the requirements

$$\frac{\partial}{\partial N_r} \Gamma_{rX}^{(1)}(N_r \to \infty, s, t) \to \mathscr{F}_n(0, s, t) , \tag{30a}$$

$$\frac{\partial}{\partial N_r} \Gamma_{rX}^{(1)}(N_r \to -\infty, s, t) \to 0 . \tag{30b}$$

Now any physically realistic solution must maintain both $\Gamma_{rX}^{(1)}$ and $\Gamma_{rF}^{(1)}$ as positive non-zero quantities; it follows that the right-hand side of (29a) must always be positive and therefore that the N_r-derivative of $\Gamma_{rX}^{(1)}$ must be a monotone increasing function whose value lies between zero and $\mathscr{F}_n(0, s, t)$ (> 0). It is also inferred that $\Gamma_{rX}^{(1)}(N \to -\infty, s, t) \to 0$, otherwise (30b) could not be satisfied (see (29a), etc.); in fact $\Gamma_{rX}^{(1)}$ must be $o(|N_r|^{-\nu_F/\nu_X})$ as $N_r \to -\infty$ in order to satisfy (30b). Similarly $\Gamma_{rX}^{(1)} - \mathscr{F}_n(0, s, t) N_r$ must be $o(|N_r|^{-\nu_X/\nu_F})$ as $N_r \to \infty$ in order to maintain boundedness of $\partial \Gamma_{rX}^{(1)}/\partial N_r$. It is convenient to define new variables

$$G(M) = \{ \mathscr{F}_n(0, s, t) \mathscr{L} \}^{-2a_r} \Gamma_{rX}^{(1)} , \tag{31a}$$

$$M = \mathscr{F}_n(0, s, t) N_r \{ \mathscr{F}_n(0, s, t) \mathscr{L} \}^{-2a_r} , \tag{31b}$$

in terms of which (29) and (30) become

$$G'' = G^{\nu_X}(G - M)^{\nu_F}; \quad G'(M \to \infty) \to 1, \ G'(M \to -\infty) \to 0 . \tag{32}$$

(A prime on G denotes differentiation with respect to M). Evidently $\Gamma_{rX}^{(1)}$ depends only parametrically on s and t, but in a quite complicated way, via the functions $\mathscr{F}_n(0, s, t)$ and $\mathscr{L}$. It is clear that all geometries with the assumed scale-factor behaviour (see paragraph containing (27)) are catered for. As $M \to -\infty$ (32) takes the form

$$G'' = (-M)^{v_F} G^{v_X} ,$$ (33)

the right-hand side being in error only by a factor which is $1 + o(|M|^{-1-v_F/v_X})$. This is an equation of the Emden–Fowler type, for which there is a unique asymptotic solution [15], namely

$$G \sim \{(v_F + 2)(v_F + v_X + 1)(v_X - 1)^{-2}\}^{1/(v_X - 1)} (-M)^{-(v_F + 2)/(v_X - 1)} ,$$ (34)

if $v_X > 1$ and $v_F \geqq 1$. (Observe that G, and hence $\Gamma_{rX}^{(1)}$, obeys the order restriction deduced above equations (31).) Matching therefore shows that the gauge factor $g_{rX}^{(1)}(\varepsilon)$ is given by

$$g_{rX}^{(1)}(\varepsilon) = \varepsilon^{1/(v_X - 1)} , \qquad v_X > 1 ;$$ (35)

(compare with $g_{eX}^{(1)}(\delta)$ in (25) and observe that matching is essential here), and a condition on the form of $c_{rX}^{(1)}(n \uparrow 0, s, t)$ is also derivable. It is clear from (35) that $c_{rX}^{(1)}(n < 0, s, t)$ must satisfy a version of (26) (with $\alpha = X$) for which all left-hand side terms of that equation are retained, while the right-hand side is proportional to $c_{rX}^{(1)v_X}[-\mathscr{F}(n, s, t)]^{v_F}$ (recall that $\mathscr{F} < 0$ in $n < 0$). This is a very difficult equation to solve and one may have to be content with the information in (35), which at least gives the order of magnitude of the oxidant concentration in D_F.

It may well happen that v_X is unity, in which case (33) is simply a linear second order equation; in such a case it can be shown that the desired solution for G has an asymptotic form

$$G \sim \text{Constant} \, (-M)^{-v_F/4} \exp\{-(-M)^{1+v_F/2}/(1 + v_F/2)\}$$ (36)

as $M \to -\infty$, $v_X = 1$, $v_F \geqq 1$. When $v_X = 1$ the left-hand side of (26) ($\alpha = X$) always vanishes in the outer limit as $\varepsilon \downarrow 0$, no matter how c_X is scaled. The conclusion that c_X is zero in the outer region D_F ($n < 0$) is supported by the evidence of (36) if zero is taken to mean exponentially small in these circumstances.

Returning to the inner region it is noted that, the left-hand side of (10) is $O(\varepsilon^{1-a_r})$ where a_r is given by (28). The first term on the right-hand side of (10) is also of this order and so it is valid to use (26) in this region if $\delta \varepsilon^{-1+a_r}$ is $o(1)$. Comparing (17) and (28), this is equivalent to finding that $\delta \varepsilon^{-1/(1+a_e)}$ is $o(1)$; since it must be true that $\delta^{1+a_e} \varepsilon^{-1}$ is $o(1)$ if the latter is true, it can be seen at once that the criterion for dropping the term in δ in the present case is reciprocal to the criterion for omitting terms in ε in the previous case (see section 3). It follows that the final term in (10) must be retained in the inner region when $\delta \varepsilon^{-1+a_r}$ or, what is equivalent, $\varepsilon \delta^{-1-a_e}$, is $O(1)$. In such circumstances a positive constant must be subtracted from the right-hand side of (29a) to give the new equation for $\Gamma_{rX}^{(1)}$, that constant being related to the hypothesised proportionality between δ and ε^{1-a_r}. Since $\partial^2 \Gamma_{rX}^{(1)}/\partial N_r^2$ must approach zero as $|N_r| \to \infty$, so as to preserve the boundedness of $\partial \Gamma_{rX}^{(1)}/\partial N_r$, it follows that $\Gamma_{rX}^{(1)}$ will behave in a pseudo-equilibrium way (compare (22)) near the edges of the inner region and therefore (from matching requirements) also in the outer wings of the flame on either side of the surface S.

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